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On Certain Trigonometrical Expansions

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1. Introduction. The work described in this paper arises from a study of the long waves in the North Sea generated by a wind field.

In this context there occurs the problem of^{1,2} how to determine a solution of the equation of HELMHOLTZ

$$(1.1) \quad \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - q^2 F = 0,$$

in the region $0 < x < \pi$, $y > 0$, which vanishes for $y \rightarrow \infty$, and which satisfies the boundary conditions

$$(1.2) \quad F = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \pi,$$

$$(1.3) \quad \cos \mu\pi \frac{\partial F}{\partial y} - \sin \mu\pi \frac{\partial F}{\partial x} + f(x) = 0 \quad \text{at} \quad y = 0.$$

It will be assumed that q and μ are given real constants with $q \geq 0$ and $0 \leq \mu \leq \frac{1}{2}$, and that $f(x)$ is a given real function. In order to simplify the discussion we shall assume that $f(x)$ is an analytic function. In many cases, however, a much less stringent condition may be sufficient.

The condition (1.3) means that at $y = 0$ the solution $F(x, y)$ has a prescribed directional derivative, the direction of which makes a positive angle $\mu\pi$ with the normal. Physically $\text{tg } \mu\pi$ is proportional to the coefficient of CORIOLIS, *i.e.* the angular velocity of the earth.

The solution $F(x, y)$ may be represented as follows

$$(1.4) \quad F(x, y) = \sum_1^\infty b_k \frac{\sin kx}{k} e^{-y\sqrt{k^2+q^2}}.$$

The coefficients b_k can be determined by means of the boundary condition (1.3) at $y = 0$ which gives

$$(1.5) \quad f(x) = \sum_1^\infty b_k \left\{ \sqrt{1 + \frac{q^2}{k^2}} \cos \mu\pi \sin kx + \sin \mu\pi \cos kx \right\}.$$

¹ Cf. G. I. TAYLOR, *Proc. Lond. Math. Soc.* (2) XX 148 (1920).

² Cf. H. LAMB, *Hydrodynamics*, 6th ed., art. 209 (1932).

(a) For $q = 0$, $\mu = 0$ this relation reduces to a sine expansion

$$(1.6) \quad f(x) = \sum_1^{\infty} b_k \sin kx.$$

In this case the coefficients b_k may be determined in the familiar way and they are of harmonic order $b_k = O(k^{-1})$.

(b) For $q = 0$, $\mu = \frac{1}{2}\pi$ (1.5) reduces to a cosine expansion

$$(1.7) \quad f(x) = \sum_1^{\infty} b_k \cos kx.$$

In this case no convergent expansion exists unless $\int_0^{\pi} f(t) dt = 0$.

(c) For $q = 0$, $0 < \mu < \frac{1}{2}$ (1.5) reduces to

$$(1.8) \quad f(x) = \sum_1^{\infty} b_k \sin (kx + \mu\pi).$$

This expansion will be the subject of the first part of this paper. The results are as follows: The expansion (1.8) is unique and b_k is of subharmonic order $b_k = O(k^{-1+2\mu})$.

In particular, we have the special case

$$(1.9) \quad 1 = \sum_1^{\infty} (-1)^{k-1} e_k(2\mu) \sin (kx + \mu\pi), \quad 0 < x < \pi,$$

where $e_k(\alpha)$ is defined by the generating function

$$(1.10) \quad \left(\frac{1+s}{1-s} \right)^{\alpha} = \sum_0^{\infty} e_k(\alpha) s^k, \quad |s| < 1.$$

We have

$$e_0 = 1, \quad e_1 = 2\alpha, \quad e_2 = 2\alpha^2, \dots$$

and for arbitrary index $k \geq 1$

$$(1.11) \quad e_k(\alpha) = (-1)^{k-1} 2\alpha F(1-k, 1+\alpha; 2; 2).$$

The coefficients b_k of (1.8) may be expressed by means of

$$b_k = \frac{1}{\pi} \int_0^{\pi} k_k(t) f(t) dt,$$

where $k_m(x)$ is the m^{th} orthogonal function with respect to the set $\sin (nx + \mu\pi)$. These functions may be determined by the generating function

$$(1.12) \quad \sum_1^{\infty} k_m(x) s^m = \left(\frac{1+s}{1-s} \operatorname{tg} \frac{x}{2} \right)^{-2\mu} \sum_1^{\infty} 2 \sin kx s^k.$$

The functions $k_m(x)$ are uniformly bounded. For all $m \geq 1$ and all x in $(0, \pi)$ we have

$$(1.13) \quad |k_m(x)| \leq 2 + 4 \sin \mu\pi.$$

We note that the function $(\operatorname{tg} \frac{1}{2}x)^{-2\mu}$ is orthogonal with respect to all functions of the set $\sin (nx + \mu\pi)$, viz.

$$(1.14) \quad \frac{1}{\pi} \int_0^\pi \left(\operatorname{tg} \frac{t}{2} \right)^{-2\mu} \sin (nt + \mu\pi) dt = \delta_{0n}$$

for $n = 0, 1, 2, \dots$. If for the function $f(x)$ of (1.8) the following condition is fulfilled:

$$\frac{1}{\pi} \int_0^\pi \left(\operatorname{tg} \frac{t}{2} \right)^{-2\mu} f(t) dt = 0,$$

then the coefficients b_k are of hyperharmonic order $b_k = O(k^{-1-2\mu})$.

(d) We shall consider the slightly more general case

$$(1.15) \quad f(x) = \sum_1^\infty b_k (\sin kx + \theta_k \cos kx), \quad 0 < x < \pi,$$

where $\theta_k = \theta + O(\theta^{-c})$, $c > 1$, for $k \rightarrow \infty$. In the case given by (1.5) we have

$$(1.16) \quad \theta_k = \left(1 + \frac{q^2}{k^2} \right)^{-\frac{1}{2}} \operatorname{tg} \mu\pi, \quad \theta = \operatorname{tg} \mu\pi.$$

Relatively little seems to be known concerning expansions of this kind. We quote the following result due to PALEY & WIENER³ If $|\theta_k| \leq c \leq 1$ and if $f(x) \in L^2$ a unique expansion of the kind (1.15) exists with $\sum_1^\infty b_k^2 < \infty$.

In section 5 it will be shown that the expansion problem (1.15) may be reduced to either an infinite system of linear equations or an integral equation of the Fredholm type.

The results are as follows: For $\theta > 0$ a convergent expansion of the kind (1.15) exists and is unique. The coefficients b_k are of subharmonic order

$$(1.17) \quad b_k = \frac{(-1)^k \beta}{k^{1-2\mu}} + O(k^{-1-2\mu}), \quad k \rightarrow \infty,$$

where β does not depend on k , and where $\mu = (1/\pi) \operatorname{arctg} \theta$.

In a way similar to the preceding case the coefficient b_k may be written in the form

$$(1.18) \quad b_k = \frac{1}{\pi} \int_0^\pi m_k(t) f(t) dt,$$

where the functions $m_k(x)$ are bounded in $(0, \pi)$ and orthogonal with respect to the set $\sin kx + \theta_k \cos kx$.

There exists a function $l_0(x)$ for which

$$(1.19) \quad \frac{1}{\pi} \int_0^\pi l_0(t) (\sin nt + \theta_n \cos nt) dt = \delta_{0n}.$$

³ *Fourier transforms in the complex plane*, Am. Math. Soc., New York, p. 100 (1943).

This function is bounded in $(0, \pi)$ with the exception of $x = 0$ where, however,

$$\left(\operatorname{tg} \frac{x}{2}\right)^{-1+2\mu} l_0(x)$$

has a finite limit.

Returning to the boundary value problem considered at the beginning we may conclude that a unique solution $F(x, y)$ exists which is continuous and bounded throughout the semi-infinite strip $0 \leq x \leq \pi$, $0 \leq y \leq \infty$. The behaviour of $F(x, y)$ at the corners $x = 0$ and $x = \pi$, $y = 0$ is determined by

$$x \rightarrow 0, \quad y \rightarrow 0, \quad F(x, y) = c_1 + c_2 r^{1+2\mu} \sin(1 + 2\mu) \left(\frac{\pi}{2} - \varphi\right) + \text{higher terms},$$

$$x \rightarrow \pi, \quad y \rightarrow 0, \quad F(x, y) = c_3 + c_4 r^{1-2\mu} \sin(1 - 2\mu) \left(\frac{\pi}{2} - \varphi\right) + \text{higher terms},$$

where (r, φ) are local polar coordinates and where c_1, c_2, c_3, c_4 are constants.

2. We shall consider at first the expansion of $f(x)$ into a trigonometrical series with a constant term

$$(2.1) \quad f(x) = \sum_0^{\infty} a_k \sin(kx + \mu\pi), \quad 0 < x < \pi,$$

with $0 < \mu < \frac{1}{2}$. The discussion of this expansion is somewhat simpler than that of the expansion (1.8). We shall require that

$$(2.2) \quad \sum_0^{\infty} |a_k| < \infty.$$

For the present purposes it is sufficient that $f(x)$ satisfies a Hölder condition of the following kind:

$$(2.3) \quad |f(x_1) - f(x_2)| < c |x_1 - x_2|^{-\gamma}$$

for any pair x_1, x_2 from the closed interval $(0, \pi)$ and where c and γ are positive constants not depending on x_1 and x_2 .

We shall show that under these assumptions the expansion (2.1) exists and is unique. Consider the analytic function

$$(2.4) \quad \phi(z) = \sum_0^{\infty} a_k e^{kz},$$

where $z = x + yi$, $0 \leq x \leq \pi$, $y \geq 0$. This function satisfies the conditions

$$(2.5) \quad \begin{aligned} \operatorname{Im} \phi &= 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \pi, \\ \operatorname{Im} e^{\mu\pi i} \phi &= 0 \quad \text{at} \quad y = 0, \end{aligned}$$

ϕ uniformly bounded in $0 \leq x \leq \pi$, $y \geq 0$. By means of the following conformal mapping the semi-infinite strip $0 \leq x \leq \pi$, $y \geq 0$ is mapped upon the upper

half plane

$$(2.6) \quad w = -\cos z, \quad w = u + vi.$$

By this the problem (2.5) is transformed into

$$(2.7) \quad \begin{aligned} \operatorname{Im} \phi &= 0 & \text{at } v &= 0, & |u| &> 1, \\ \operatorname{Im} e^{\mu\pi i} \phi &= g(u) & \text{at } v &= 0, & |u| &< 1, \end{aligned}$$

ϕ uniformly bounded in $-\infty < u < \infty$, $v \geq 0$, where $g(-\cos x) \equiv f(x)$. The problem (2.7) is a Hilbert problem for a half plane.⁴ Its solution may be represented as follows by a Cauchy integral

$$(2.8) \quad \phi(w) = -\frac{1}{\pi} (w-1)^{1-\mu} (w+1)^{\mu} \int_{-1}^1 (1-t)^{-1+\mu} (1+t)^{-\mu} \frac{g(t)}{t-w} dt.$$

This expression has at $w = \pm 1$ the finite limit $g(\pm 1) \operatorname{cosec} \mu\pi$.

Any other solution of (2.7) differs from (2.8) by a solution of the corresponding homogeneous problem. The latter is solved by a linear combination of the functions

$$(2.9) \quad (w+1)^{-\nu_1+\mu} (w-1)^{\nu_2-\mu}$$

where ν_1 and ν_2 are integers with $\nu_1 \geq \nu_2$. Since these functions are infinite either at $w = -1$ or $w = 1$ they do not comply with the last condition of (2.7). The problem (2.7) has therefore the unique solution (2.8).

From (2.8) we may derive

$$(2.10) \quad \begin{aligned} \sum_1^{\infty} a_k \sin kx &= \operatorname{Im} \phi(x) \\ &= (1 + \cos x) \left(\operatorname{tg} \frac{x}{2} \right)^{2\mu} \frac{1}{\pi} \int_0^{\pi} \left(\operatorname{tg} \frac{t}{2} \right)^{1-2\mu} \frac{f(t) - f(x)}{\cos x - \cos t} dt. \end{aligned}$$

If $f(x)$ is analytic, from the right-hand side of (2.9) an asymptotic series for a_k in negative powers of k may be obtained. If $f(x)$ has a restricted number of derivatives a finite asymptotic series with a remainder is obtained. The first few terms of this asymptotic expression are determined by the behaviour of $\operatorname{Im} \phi(x)$ at $x = 0$ and at $x = \pi$ only. By means of a familiar technique⁵ we obtain

$$(2.11) \quad a_k = \frac{2^{1-2\mu} \varphi(0)}{\Gamma(-2\mu)} k^{-1-2\mu} + (-1)^{k-1} \frac{2^{2\mu-1} \varphi(\pi)}{\Gamma(2\mu-2)} k^{-3+2\mu} + O(k^{-3-2\mu}),$$

where

$$(2.12) \quad \varphi(x) = \frac{1}{\pi} \int_0^{\pi} \left(\operatorname{tg} \frac{t}{2} \right)^{1-2\mu} \frac{f(t) - f(x)}{\cos x - \cos t} dt.$$

⁴ Cf. MUSKHELISHVILI, *Singular integral equations*, Groningen (1953), Ch. 10.

⁵ C. G. LEKKERKERKER & A. H. M. LEVELT, *Asymptotic Expansion of a certain Fourier coefficient*, Report ZW1957-012, Math. Centrum.

We shall show that the expansion (2.1) is no longer unique if the condition (2.2) is dropped. We may, however, still require that $c_k \rightarrow 0$. This follows easily from the Hilbert problem considered above. To the particular solution (2.7) one may always add a constant multiple of $(w+1)^{-1+\mu}(w-1)^{-\mu}$. Since the resulting singularities of $\phi(z)$ at $z=0$ and $z=\pi$ remain integrable the coefficient c_k tends to zero.

An alternative, more elementary way is as follows: Consider the Taylor expansion

$$(2.13) \quad \left(\frac{1+s}{1-s}\right)^\alpha = \sum_0^\infty e_k(\alpha) s^k, \quad |s| < 1.$$

The first few coefficients are

$$e_0 = 1, \quad e_1 = 2\alpha, \quad e_2 = 2\alpha^2, \quad e_3 = \frac{4\alpha^3 + 2\alpha}{3}.$$

They satisfy the recurrence relation

$$(2.14) \quad ke_k = 2\alpha e_{k-1} + (k-2)e_{k-2}.$$

Hence $k! e_k$ is a polynomial in 2α with integer coefficients. We have, of course, for e_k the expression

$$(2.15) \quad e_k(\alpha) = \frac{1}{2\pi i} \oint \left(\frac{1+s}{1-s}\right)^\alpha \frac{ds}{s^{k+1}}.$$

From this we may derive

$$(2.16) \quad e_k(\alpha) = (-1)^{k-1} 2\alpha F(1-k, 1+\alpha; 2; 2), \quad k \geq 1.$$

Either from (2.15) or from (2.16) the following asymptotic expansion may be obtained:

$$(2.17) \quad e_k(\alpha) = \frac{2^\alpha}{\Gamma(\alpha)k^{1-\alpha}} + \frac{(-1)^k 2^{-\alpha}}{\Gamma(-\alpha)k^{1+\alpha}} + \frac{\alpha}{12} \frac{2^\alpha}{\Gamma(\alpha-2)k^{3-\alpha}} + O(k^{-3-\alpha}),$$

where $0 < \alpha < 1$.

If in (2.13), with $-1 < \alpha < 1$ and $0 < x < \pi$, $s = e^{ix}$ is substituted we obtain, separating real and imaginary parts,

$$(2.18) \quad \cos \frac{\alpha\pi}{2} \left(\operatorname{tg} \frac{x}{2}\right)^{-\alpha} = \sum_0^\infty e_k(\alpha) \cos kx,$$

$$(2.19) \quad \sin \frac{\alpha\pi}{2} \left(\operatorname{tg} \frac{x}{2}\right)^{-\alpha} = \sum_1^\infty e_k(\alpha) \sin kx.$$

In particular, we obtain for $\alpha = -2\mu$

$$(2.20) \quad 0 = \sum_0^\infty e_k(-2\mu) \sin(kx + \mu\pi).$$

This is an expansion of zero into a slowly converging trigonometric series for which the condition (2.2) is violated.

We return now to the original expansion (2.1). For the coefficient a_0 we find easily by means of $a_0 = \lim_{w \rightarrow \infty} \phi(w)$ for $w \rightarrow \infty$

$$(2.21) \quad a_0 = \frac{1}{\pi} \int_0^\pi \left(\operatorname{tg} \frac{t}{2} \right)^{1-2\mu} f(t) dt.$$

By combining the expansions (2.1) and (2.20) an expansion of $f(x)$ into a trigonometric series without constant term is obtained.

$$(2.22) \quad f(x) = \sum_1^\infty b_k \sin(kx + \mu\pi),$$

where

$$(2.23) \quad b_k = a_k - a_0 e_k(-2\mu).$$

From (2.11) and (2.17) we obtain the asymptotic expression

$$(2.24) \quad b_k = \frac{(-1)^{k-1} 2^{2\mu}}{\Gamma(2\mu)} \frac{A}{k^{1-2\mu}} + \frac{2^{-2\mu}}{\Gamma(-2\mu)} \frac{B}{k^{1+2\mu}} + \frac{(-1)^{k-1} 2^{2\mu-2}}{\Gamma(2\mu-2)} \frac{C}{k^{3-2\mu}} + O(k^{-3-2\mu}),$$

where

$$(2.25) \quad A = a_0, \quad B = 2\varphi(0) - A, \quad C = 2\varphi(\pi) + \frac{2}{3}\mu A.$$

3. The biorthogonal functions. We shall derive a set of integrable functions $h_m(x)$, $m = 0, 1, 2, \dots$, which are orthogonal with respect to the set $\sin(nx + \mu\pi)$, $n = 0, 1, 2, \dots$. Thus we require that

$$(3.1) \quad \frac{1}{\pi} \int_0^\pi h_m(t) \sin(nt + \mu\pi) dt = \delta_{mn}$$

for $m, n \geq 0$.

From (2.18) and (2.19) we obtain with $\alpha = -1 + 2\mu$

$$(3.2) \quad \frac{1}{\pi} \int_0^\pi \left(\operatorname{tg} \frac{t}{2} \right)^{1-2\mu} \cos kt dt = \frac{e_k(-1 + 2\mu)}{2 \sin \mu\pi}, \quad k \geq 1,$$

$$(3.3) \quad \frac{1}{\pi} \int_0^\pi \left(\operatorname{tg} \frac{t}{2} \right)^{1-2\mu} \sin kt dt = \frac{-e_k(-1 + 2\mu)}{2 \cos \mu\pi}, \quad k \geq 1.$$

Hence it follows that

$$(3.4) \quad \frac{1}{\pi} \int_0^\pi \left(\operatorname{tg} \frac{t}{2} \right)^{1-2\mu} \sin(nt + \mu\pi) dt = 0$$

for $n \geq 1$. Thus we find with the proper normalisation

$$(3.5) \quad h_0(x) = \left(\operatorname{tg} \frac{x}{2} \right)^{1-2\mu}.$$

From (3.2) and (3.3) we obtain for $m < n$

$$(3.6) \quad \frac{1}{\pi} \int_0^\pi \left(\operatorname{tg} \frac{t}{2} \right)^{1-2\mu} \cos mt \sin (nt + \mu\pi) dt = 0,$$

so that $h_m(x)$ can be represented by the product of $h_0(x)$ and a finite cosine series. Without difficulty we obtain

$$(3.7) \quad h_m(x) = \left(\operatorname{tg} \frac{x}{2} \right)^{1-2\mu} \sum_0^m \epsilon_j e_{m-j} (1 - 2\mu) \cos jx,$$

where $\epsilon_j = 1$ for $j = 0$ and $\epsilon_j = 2$ for $j \geq 1$.

We note that the functions $h_m(x)$ may be derived from the generating function

$$(3.8) \quad \sum_0^\infty h_m(x) s^m = \left(\frac{1+s}{1-s} \operatorname{tg} \frac{x}{2} \right)^{1-2\mu} \frac{1-s^2}{1-2s \cos x + s^2}.$$

Finally we mention the asymptotic relation

$$(3.9) \quad h_m(x) = 2 \sin (mx + \mu\pi) + O(m^{-1-2\mu})$$

for $0 < x < \pi$, $m \rightarrow \infty$.

The coefficients a_k of the rapidly convergent expansion

$$(3.10) \quad f(x) = \sum_0^\infty a_k \sin (kx + \mu\pi)$$

may be expressed by means of the biorthogonal functions $h_m(x)$ in the following way:

$$(3.11) \quad a_k = \frac{1}{\pi} \int_0^\pi h_k(t) f(t) dt.$$

From the biorthogonal set $h_m(x)$, $m = 0, 1, 2, \dots$, we may derive a reduced biorthogonal set of functions $k_m(x)$, $m = 1, 2, \dots$, which are bounded in the closed interval $(0, \pi)$. We shall define

$$(3.12) \quad k_m(x) = h_m(x) - e_m(-2\mu) h_0(x), \quad m \geq 1.$$

By means of (3.8) we obtain for $k_m(x)$ the generating function

$$(3.13) \quad k(x, s) \equiv \sum_1^\infty k_m(x) s^m = \left(\frac{1+s}{1-s} \operatorname{tg} \frac{x}{2} \right)^{-2\mu} \frac{2s \sin x}{1-2s \cos x + s^2}.$$

This gives the following explicit representation

$$(3.14) \quad k_m(x) = \left(\operatorname{tg} \frac{x}{2} \right)^{-2\mu} \sum_1^m 2e_{m-j}(-2\mu) \sin jx.$$

The coefficients b_k of the slowly convergent expansion

$$(3.15) \quad f(x) = \sum_1^\infty b_k \sin (kx + \mu\pi)$$

may be expressed by means of the reduced biorthogonal functions. In view of (2.23) and (3.12) we have

$$(3.16) \quad b_k = \frac{1}{\pi} \int_0^\pi k_k(t) f(t) dt.$$

Finally we shall show that the functions $k_m(x)$ admit the following uniform bound:

$$(3.17) \quad |k_m(x)| \leq 2 + 4 \sin \mu\pi.$$

The proof is as follows: From (3.13) we obtain

$$(3.18) \quad k_m(x) = \frac{1}{2\pi i} \oint k(x, s) s^{-m-1} ds.$$

The function $k(x, s)$ has in the complex s -plane poles at $s = e^{\pm xi}$ and branch points at $s = \pm 1$. The contribution of the poles equals $2 \sin (mx + \mu\pi)$. The contributions of the branch points are equal to two integrals which can be determined in the following way:

$$k_m(x) = 2 \sin (mx + \mu\pi) - \left(\sin \frac{x}{2} \right)^{1-2\mu} \left(\cos \frac{x}{2} \right)^{1+2\mu} \frac{4 \sin 2\mu\pi}{\pi} \cdot \{ \alpha(\mu, x) + (-1)^m \alpha(-\mu, \pi - x) \}$$

where

$$\alpha(\mu, x) = \int_1^\infty \frac{(t-1)^{2\mu} (t+1)^{2\mu}}{t^2 - 2t \cos x + 1} \frac{dt}{t^m}.$$

The inequality (3.17) follows easily from the following two estimates:

$$\alpha(\mu, x) \leq 2^{-2\mu} \int_0^\infty \frac{u^{2\mu} du}{u^2 + 2(1 - \cos x)} = \left(\sin \frac{x}{2} \right)^{-1+2\mu} \frac{\pi}{4 \cos \mu\pi},$$

$$\alpha(-\mu, \pi - x) \leq 2^{2\mu} \int_0^\infty \frac{u^{-2\mu} du}{u^2 + 2(1 + \cos x)} = \left(\cos \frac{x}{2} \right)^{-1-2\mu} \frac{\pi}{4 \cos \mu\pi}.$$

4. Analytic continuation. The results of the preceding two sections may easily be extended to complex values of μ . Without loss of generality we may suppose that $0 \leq \operatorname{Re} \mu \leq \frac{1}{2}$. Since the case $0 < \operatorname{Re} \mu < \frac{1}{2}$ does not lead to special considerations we restrict ourselves to the cases $\operatorname{Re} \mu = 0$ and $\operatorname{Re} \mu = \frac{1}{2}$.

(a) $\operatorname{Re} \mu = \frac{1}{2}$.

We may put

$$(4.1) \quad \begin{cases} \mu = \frac{1}{2} + \frac{1}{2\pi i} \ln \frac{\theta - 1}{\theta + 1}, & \theta > 1, \\ \operatorname{tg} \mu\pi = \theta i. \end{cases}$$

The rapidly convergent expansion (2.1) is possible. We have $a_k = O(k^{-2})$.

An expansion without constant term is generally not possible since $e_k(2\mu)$ does not have a limit for $k \rightarrow \infty$. In this case the full set of biorthogonal functions $h_m(x)$, $m = 0, 1, 2, \dots$, exists.

(b) $\operatorname{Re} \mu = 0$.

We may put

$$(4.2) \quad \begin{cases} \mu = \frac{1}{2\pi i} \ln \frac{1-\theta}{1+\theta}, & -1 < \theta < 1, \\ \operatorname{tg} \mu\pi = \theta i. \end{cases}$$

The slowly convergent expansion (2.22) is possible. The asymptotic behaviour of b_k is, according to (2.24),

$$(4.3) \quad b_k = \frac{(-1)^{k-1} 2^{2\mu}}{\Gamma(2\mu)} \frac{A}{k^{1-2\mu}} + \frac{2^{-2\mu}}{\Gamma(-2\mu)} \frac{B}{k^{1+2\mu}} + O(k^{-3}).$$

Hence b_k contains two harmonic terms. Nothing regarding speed of convergence can be gained by including a single constant term in the trigonometric expansion since both A and B must vanish in order to get something better than harmonic convergence. The explicit expressions of A and B follow from (2.25). They are

$$(4.4) \quad A = \frac{1}{\pi} \int_0^\pi \left(\operatorname{tg} \frac{t}{2} \right)^{1-2\mu} \{f(t) - f(\pi)\} dt + \frac{f(\pi)}{\sin \mu\pi},$$

$$(4.5) \quad B = \frac{1}{\pi} \int_0^\pi \left(\operatorname{tg} \frac{t}{2} \right)^{-1-2\mu} \{f(t) - f(0)\} dt - \frac{f(0)}{\sin \mu\pi}.$$

These expressions are valid in the strip $-\frac{1}{2} < \operatorname{Re} \mu < \frac{1}{2}$. We note that for $\operatorname{Re} \mu > 0$ the expression (4.3) may be reduced to the simpler form (2.21). For $\operatorname{Re} \mu < 0$ the expression (4.4) may be simplified likewise. In this case the set of biorthogonal functions $k_m(x)$, $m = 1, 2, \dots$, should be used since the functions $h_m(x)$ have a non-integrable singularity at $x = \pi$.

5. In this section we shall consider the more general expansion

$$(5.1) \quad f(x) = \sum_1^\infty b_k (\sin kx + \theta_k \cos kx), \quad 0 < x < \pi,$$

where $\theta_k = \theta + O(\theta^{-c})$, $c > 1$ for $k \rightarrow \infty$. Without loss of generality we may assume that $\theta \geq 0$. We shall show that this expansion problem may be reduced to an integral equation. Two possible ways present themselves. If the auxiliary function

$$s(x) = \sum_1^\infty b_k \sin kx$$

is introduced the relation (5.1) may be written as

$$(5.2) \quad s(x) + \theta S\{s(x)\} + K\{s(x)\} = f(x),$$

where the operators S and K are given by

$$(5.3) \quad S\{s(x)\} = \sum_1^{\infty} b_k \cos kx,$$

$$(5.4) \quad K\{s(x)\} = \sum_1^{\infty} b_k(\theta_k - \theta) \cos kx.$$

We have for S , which transforms the sine series into the conjugate cosine series,

$$(5.5) \quad S\{s(x)\} = \frac{1}{\pi} \text{P. V.} \int_0^{\pi} \frac{\sin t s(t)}{\cos x - \cos t} dt.$$

For K we have

$$(5.6) \quad K\{s(x)\} = \frac{1}{\pi} \int_0^{\pi} K(x, t)s(t) dt,$$

with

$$(5.7) \quad K(x, t) = 2 \sum_1^{\infty} (\theta_k - \theta) \cos kx \sin kt.$$

Hence (5.2) represents a singular integral equation with a singular operator S and a Fredholm operator K . This integral equation may be solved by the method described in MUSKHELISHVILI'S book.⁶ This method consists in the reduction of (5.2) to an ordinary integral equation of the Fredholm type.

In the following more direct way a Fredholm equation may be derived from (5.1) immediately: If the auxiliary function

$$(5.8) \quad \psi(x) = \sum_1^{\infty} b_k(\sin kx + \theta \cos kx)$$

is introduced (5.1) may be written as

$$(5.9) \quad \psi(x) + L\{\psi(x)\} = f(x),$$

where

$$(5.10) \quad L\{\psi(x)\} = \sum_1^{\infty} b_k(\theta_k - \theta) \cos kx.$$

If we put $\theta = \text{tg } \mu\pi$ with $0 \leq \mu \leq \frac{1}{2}$ from (5.8) the following expression for b_k is obtained:

$$(5.11) \quad b_k = \frac{\cos \mu\pi}{\pi} \int_0^{\pi} k_k(t)\psi(t) dt.$$

Substitution into (5.10) gives

$$(5.12) \quad L\{\psi(x)\} = \frac{1}{\pi} \int_0^{\pi} L(x, t)\psi(t) dt,$$

⁶ See footnote 4.

with

$$(5.13) \quad L(x, t) = \cos \mu\pi \sum_1^{\infty} (\theta_k - \theta) \cos kx k_k(t).$$

In view of (3.17) L is a Fredholm operator with a uniformly bounded kernel. Hence (5.9) represents an ordinary Fredholm equation.

According to the familiar Fredholm theory, from (5.9) a unique and bounded solution $\psi(x)$ may be obtained. In exceptional cases, however, a certain condition should be imposed upon $f(x)$. From (2.24) we obtain for b_k the asymptotic expression

$$(5.14) \quad b_k = \frac{(-1)^k \beta}{k^{1-2\mu}} + O(k^{-1-2\mu}),$$

where β is a constant.

By way of illustration the following special case will be considered:

$$\theta_1 = \theta + \lambda, \quad \theta_k = \theta, \quad k \geq 2,$$

where λ is variable. The Fredholm equation (5.9) reduces to

$$\psi(x) + \cos x \frac{\lambda \cos \mu\pi}{\pi} \int_0^\pi k_1(t) \psi(t) dt = f(x).$$

Its solution is

$$\psi(x) = f(x) + a \cos x,$$

where

$$a(4\lambda\mu^2 + \theta) = -\frac{\lambda \sin \mu\pi}{\pi} \int_0^\pi k_1(t) f(t) dt.$$

Thus a solution is obtained unless $4\lambda\mu^2 + \theta = 0$. In the eigenvalue case $\lambda = -\theta/4\mu^2$ the following condition must be imposed upon $f(x)$:

$$\int_0^\pi k_1(t) f(t) dt = 0.$$

If to the expansion (5.1) a constant is added the rate of convergence may be improved. In a similar way as before we find that the rapidly convergent expansion

$$(5.15) \quad f(x) = a_0 + \sum_1^{\infty} a_k (\sin kx + \theta_k \cos kx), \quad 0 < x < \pi,$$

is generally possible, and that

$$(5.16) \quad a_k = \frac{\alpha}{k^{1+2\mu}} + O(k^{-3+2\mu}),$$

where α is a constant.

Next we turn our attention to the biorthogonal set $l_m(x)$, $m = 0, 1, 2, \dots$, with

$$(5.17) \quad \frac{1}{\pi} \int_0^\pi l_m(t)(\sin nt + \theta_n \cos nt) dt = \delta_{mn}$$

for $m, n \geq 0$. The functions $l_m(x)$ can be obtained in the following way: Let us assume for simplicity that $\theta_k \neq 0$. Consider the slowly convergent expansion

$$(5.18) \quad \cos mx = \sum_1^\infty l_{m,k}(\sin kx + \theta_k^{-1} \cos kx),$$

where $m = 0, 1, 2, \dots$. From this expansion $l_m(x)$ is obtained according to

$$(5.19) \quad l_m(x) = \epsilon_m \theta_m^{-1} \sum_1^\infty l_{m,k} \sin kx,$$

where $\theta_0 = 1$. By comparing the sine and cosine coefficients of $l_m(x)$ the orthogonality relation (5.17) may easily be verified.

By means of the biorthogonal functions $l_m(x)$ the coefficients a_k of (5.15) may be expressed by

$$(5.20) \quad a_k = \frac{1}{\pi} \int_0^\pi l_k(t)f(t) dt, \quad k \geq 0.$$

The coefficients b_k of (5.1) may be expressed in a similar way. The reduced biorthogonal set $m_m(x)$, $m = 1, 2, \dots$, where $m_m(x)$ is the generalisation of $k_m(x)$, is defined by

$$(5.21) \quad m_m(x) = l_m(x) + \lambda_m l_0(x),$$

where

$$(5.22) \quad 1 = \sum_1^\infty \lambda_k(\sin kx + \theta_k \cos kx).$$

It is obvious that

$$(5.23) \quad b_k = \frac{1}{\pi} \int_0^\pi m_k(t)f(t) dt, \quad k \geq 1.$$

We shall, however, show that the functions $m_m(x)$ are bounded in the closed interval $(0, \pi)$. From (5.14) and (5.23), (5.21) we obtain

$$(5.24) \quad b_k = \frac{(-1)^k \beta_0}{k^{1-2\mu}} \int_0^\pi l_0(t)f(t) dt + O(k^{-1-2\mu}),$$

where β_0 does not depend on $f(x)$. The function $m_m(x)$ has the sine expansion

$$(5.25) \quad m_m(x) = \sum_1^\infty (2\theta_m^{-1} l_{m,k} + \lambda_m l_{0,k}) \sin kx.$$

We shall now prove the assertion by showing that the convergence of (5.25) is of hyperharmonic order. We may apply the asymptotic expression (5.24),

however, with θ_k replaced by θ_k^{-1} . If $l'_0(x)$ is orthogonal with respect to the set $\sin nt + \theta_n^{-1} \cos nt$, $n \geq 1$, then we obtain from (5.24)

$$(5.26) \quad 2\theta_m^{-1}l_{n,k} + \lambda_m l_{0,k} = \frac{(-1)^k \beta'_0}{k^{2\mu}} \int_0^\pi l'_0(t) \{2\theta_m^{-1} \cos mt + \lambda_m\} dt + O(k^{-2+2\mu}).$$

In view of the cosine expansion of $l'_0(x)$,

$$l'_0(x) = \sum_1^\infty \lambda_k \sin kx = 1 - \sum_1^\infty \lambda_k \cos kx,$$

the integral in the right-hand side of (5.26) vanishes. Thus the convergence of (5.25) is of order $k^{-2+2\mu}$.

The zeroth biorthogonal function $l_0(x)$ has a singularity similar to that of $h_0(x) = (\operatorname{tg} (x/2))^{1-2\mu}$. This follows easily from expansion (5.19). We have

$$(5.27) \quad l_0(x) = \sum_1^\infty l_{0,k} \sin kx.$$

As in (5.26) we find

$$(5.28) \quad l_{0,k} = \frac{(-1)^k \beta''_0}{k^{2\mu}} + O(k^{-2+2\mu})$$

so that $(\pi - x)^{1-2\mu} l_0(x)$ takes a non-zero value for $x \rightarrow \pi$.

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